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Loop groups, Kaluza-Klein reduction and M-theory

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Abstract: We show that the data of a principal G-bundle over a principal circle bundle is equivalent to that of a $\widehat{LG} \stackrel{\text{def}}{=} \mathrm{U}(1) \ltimes LG$-bundle over the base of the circle bundle. We apply this to the Kaluza-Klein reduction of M-theory to IIA and show that certain generalized characteristic classes of the loop group bundle encode the Bianchi identities of the antisymmetric tensor fields of IIA supergravity. We further show that the low dimensional characteristic classes of the central extension of the loop group encode the Bianchi identities of massive IIA, thereby adding support to the conjectures of [1].

Keywords: Superstrings and Heterotic Strings, String Duality, M-Theory, Differential and Algebraic Geometry.

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1. Introduction and summary

It has long been established that IIA supergravity is the dimensional reduction of the unique theory of supergravity in eleven dimensions. We have learned in the last decade, however, that eleven dimensional supergravity is the low energy limit of a still mysterious theory termed M-theory and that the Kaluza-Klein reduction of this theory is the ten dimensional type IIA superstring. Furthermore, it has been understood that the charges and fluxes associated with antisymmetric tensor fields in these supergravity theories obey certain quantization conditions which can differ from the usual Dirac quantization condition [2, 3]. Often, it turns out that one can find geometric objects that model these 'quantized' fluxes. The relevant object for M-theory turns out to be somewhat complicated [4], but an essential ingredient is an E_8-bundle that models an element in $H^4(M,\mathbb{Z})$.[1] This E_8-bundle was first introduced in [2] and was motivated by the appearance of an E_8 gauge symmetry on boundaries in M-theory as described in [5, 6].

One can now ask, what is the relation of the quantization condition of the M-theory 4-form to that of the RR and NS forms in type IIA? The local relationship between the classical p-form fields and gauge invariances in M-theory and those of type IIA supergravity has long been understood as that of Kaluza-Klein reduction. However, the relationship between their quantization conditions is a subtle issue. For example, the Bianchi identity of type IIA supergravity, $dG_4 + G_2 \wedge H = 0$, already shows that in the presence of nonzero H the field G_4 is not a closed form. Thus, unlike the 4-form in M-theory, integral cohomology does not seem to be an appropriate context for understanding its quantization condition. Indeed, the description of the quantization condition of RR-fluxes in the type IIA superstring at vanishingly small g_s has been related to (twisted) K-theory [3] rather than ordinary integral cohomology.

Given the duality between M-theory compactified on a circle of radius $R_{11} \propto g_s$ and IIA string theory with coupling g_s, we can rephrase the above question differently. From the perspective of type IIA string theory, we wish to understand the finite g_s behavior of the quantization condition of type IIA fluxes. On the other hand, from the M-theory perspective we wish to understand how the E_8-bundle data associated with the quantization of the M-theory 4-form relates to the quantization condition of its Kaluza-Klein descendants when we take the M-theory circle to be very small.

The compatibility of K-theory in IIA and the E_8-bundle picture in M-theory was first examined in [7] by comparing topological flux contributions to the partition functions of M-theory and IIA on compact manifolds related by dimensional reduction. These comparisons presupposed vanishing H flux (equivalently, M-theory four forms which could be pulled back from IIA) and also $G_0 = 0$ as no framework exists for incorporating nonzero G_0 into M-theory data. It was established there that certain subtle topological phases in these partition functions agreed and that the quantization conditions associated with E_8 gauge bundles in M-theory were, in this sense, compatible with the K-theoretic quantization of fluxes IIA. However, their calculation indicated that there did *not* seem to exist a 1-1 correspondence between quantized flux configurations in M-theory and IIA. Rather, it was only after performing a *sum* over configurations in the partition functions that agreement was observed. It was therefore suggested that the relationship between the antisymmetric tensor fields in M-theory and IIA must be understood as a *quantum* equivalence.

Here, we take a very modest step towards better understanding this relationship by describing how loop groups of E_8 can allow one to organize and generalize the Kaluza-Klein reduction of the quantization data in M-theory to include all nontrivial RR and NS fluxes, even $G_0 \neq 0$. To do this, we will consider in detail the Kaluza-Klein reduction of the E_8-

[1]In fact, maps to any model for $K(\mathbb{Z},4)$ will do in the construction of [4]. E_8 bundles are satisfyingly geometrical, however, and they will be the focus of this paper. It is curious to note that one such alternate model of $K(\mathbb{Z},4)$ in low dimensions is $\widetilde{LE}_8^{k=1}$ bundles, where $\widetilde{LE}_8^{k=1}$ is the universal central extension of the loop group of E_8. $\widetilde{LE}_8^{k=1}$ appears in the world sheet description of the heterotic string which is dual to M-theory on manifolds with boundary [5, 6].

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bundle data and certain generalizations suggested by this process. It was conjectured [8] that the resulting data might be understood in terms of bundles of a loop group of E_8 and that such a description might be related to the K-theory description in a more transparent way, perhaps along the lines suggested in [9]. This idea was first explored in [1] and later in [10, 11], where it was suggested that, for trivial M-theory circle bundles, the dimensional reduction of the E_8-bundle in M-theory would be a LE_8-bundle over the ten dimensional base (related ideas were further developed in [12 − 16]). Here, LE_8 is the free loop group of E_8 defined as maps from $\mathbf{S}^1$ into E_8 with pointwise multiplication. However, we will find that this conjecture must be modified in order to include the case of a nontrivial M-theory principal circle bundle. We will establish a correspondence between E_8-bundles in M-theory and $\widehat{LE_8}$-bundles in IIA, where $\widehat{LE_8}$ is a slightly modified version of the loop group given by $\widehat{LE_8} \stackrel{\text{def}}{=} \mathrm{U}(1) \ltimes LE_8$.

It was further conjectured in [1] that massive IIA supergravity [17] would be related to the centrally extended loop group $\widetilde{LE}_8^k$. If this could be verified and an appropriate loop group generalization of the η invariants used in the work of [7, 18] defined, one might be able to extend their results to $G_0 \neq 0$ by adding to the M-theory partition function sectors corresponding to the $\widetilde{LE}_8^k$ quantization of massive IIA for all $k = G_0$.[2] While such a task is far beyond the scope of this work, we will show that the characteristic classes of $\widetilde{LE}_8^k$ bundles can reproduce the Bianchi identities of massive IIA (thereby explaining many of the results of [16]). A further simple consequence [16] of the $\widetilde{LE}_8^k$ quantization is that D6-brane charge is valued in $\mathbb{Z}_k$ in massive IIA. We describe how this result can be understood via a Stückelberg mechanism and how it is verified in the work of [21] on Calabi-Yau compactifications of massive IIA. Finally, using this correspondence, we will describe a simple example of the puzzles that arise when one attempts to compare the the K-theory classification of fluxes directly with the loop group bundle picture.

The outline of our paper is as follows. In section 2, we will show directly how to construct this $\widehat{LE_8}$ bundle as a special case of a more general construction which relates G bundles on a principal circle bundle Y to $\widehat{LG}$ bundles on its base X. In section 3, we will discuss the characteristic classes for this bundle. We will show that the topological data of the $\widehat{LE_8}$ bundle in low dimensions is faithfully encoded in a pair of classes, one in ordinary cohomology and the other in a generalized cohomology theory inspired by the Gysin sequence. In particular, we will see that these classes necessarily obey the Bianchi identities of IIA supergravity. We will also show that a similar relationship holds between the characteristic classes of $\widetilde{LE}_8^k$ bundles and the Bianchi identities of massive IIA supergravity. In order to establish these results, it will be necessary both to compute the cohomology of the classifying space of $\widehat{LE_8}$ bundles and to explore the generalized cohomology theory alluded to earlier. This will be relegated to section 4 and the appendices to spare the reader who wishes to avoid such details. Finally, we consider some interesting examples and comment on the relation between the loop group bundles explored in this paper and the K-theory classification given in [3] which was fruitfully exploited in [7, 18]. This and other issues will be the subject of section 5.

[2]For an alternative viewpoint on incorporating massive IIA backgrounds in M-theory see [18 − 20].

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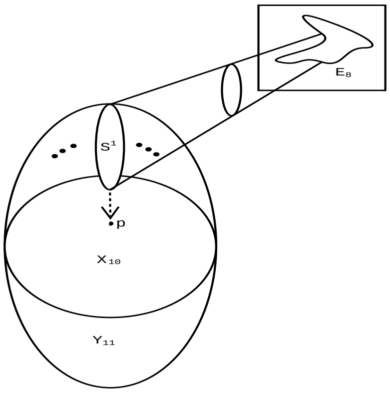

Figure 1: The basic setup.

Previous work relating to the appearance of Kac-Moody symmetries in Kaluza-Klein reduction has appeared in [22, 23].

2. $\widehat{LG}$ bundles and Kaluza-Klein reduction

The basic setup we will consider is diagrammed in (2.1). We have an $(n+1)$-manifold Y which is a U(1) *principal* bundle[3] over a simply connected n-manifold X, and a G-bundle E over Y. In the specific case of the reduction of M-theory to type IIA string theory, Y is eleven dimensional, X is ten dimensional, and we consider $G = E_8$ as in figure 1.

$$\begin{tikzcd} G \arrow[r] & E \arrow[d, "\pi"] \\ \mathrm{U}(1) \arrow[r] & Y \arrow[d, "\rho"] \\ & X \end{tikzcd} \quad . \tag{2.1}$$

Before giving the correct construction of the $\widehat{LG}$ bundle, let us see where the attempt to construct a LG bundle goes wrong. Over any point $p \in X$, $\rho^{-1}(p)$ is a circle, $\mathbf{S}^{\mathbf{1}}_p \subset Y$. As long as G is connected, any G-bundle over a circle is trivial and has global sections. Thus, for each p we consider the space of sections, $\Gamma(\rho^{-1}(p), E) = \Gamma(\mathbf{S}^{\mathbf{1}}_p, E)$. The triviality of the bundle over $\mathbf{S}^{\mathbf{1}}_p$ allows us to think of this as maps from $\mathbf{S}^{\mathbf{1}}_p$ to G. Taking the union

[3] More generally, we could have a $\mathbf{S}^1$ fiber bundle with structure group Diff$(\mathbf{S}^1)$. Much of the following construction is identical. However, in the end, instead of a principal $\widehat{LG}$-bundle we would obtain a $\mathbf{S}^1 \times LG$ bundle with structure group Diff$(\mathbf{S}^1) \ltimes LG$. We do not know how to characterize such bundles. One cannot help but notice the relation of this sort of symmetry to that in CFT. We do not know this significance of this.

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of these spaces for all $p \in X$, one might think that we have formed a principal LG bundle, as the free loop group is given by $\mathrm{Maps}(\mathbf{S}^1 \to G)$. However, a principal LG bundle also possesses a free left LG action. As the multiplication on LG is pointwise, in order to define a global LG action, we would need to identify the points of the $\mathbf{S}^1_p$ for all $p \in X$ with the points of the single circle in the group LG. This immediately implies that there exists a global section of the original $\mathbf{S}^1$ bundle. This is only possible when Y is a trivial bundle, i.e., $Y \cong X \times \mathbf{S}^1$. In fact, if we choose any global section of Y, we can use the U(1) action on Y to give an identification of each $\mathbf{S}^1_p$ with the original group U(1) where the section defines an identity. So, in essence, what we lack is a choice of an identity on each $\mathbf{S}^1_p$.[4]

Lacking this identity in general, we will simply adjoin one to the construction.[5] In other words, over each point $p \in X$, we take the space $\mathbf{S}^1_p \times \Gamma(\mathbf{S}^1_p, E)$ where the first factor corresponds to all possible choices of identity. This is topologically $\mathbf{S}^1 \times LG$. Taking the union for all points p gives a bundle which we will denote $\widehat{E}$.

We now show that this is a principal bundle. Given the point $s \in \mathbf{S}^1_p$, the U(1) action on Y gives us an identification of $\mathbf{S}^1_p$ with U(1). We use this to let LG act on $\Gamma(\mathbf{S}^1_p, E)$. Finally, there is a U(1) action on the $\mathbf{S}^1$ part of the fiber of $\widehat{E}$ which acts exactly as does the U(1) action on the fiber of Y. This group action is, in fact, a semidirect product, $\widehat{LG} = \mathrm{U}(1) \ltimes LG$. To see this we note that for each point $s \in \mathbf{S}^1_p$, the U(1) action on Y gives us a particular identification of $\mathbf{S}^1_p$ with U(1) such that s is mapped to the identity element in U(1). Using this map, we can associate to any $t \in \mathbf{S}^1_p$ an angle θ by the relation $t = s + \theta \in \mathbf{S}^1_p$, where we denote the action of U(1) on $\mathbf{S}^1_p$ by addition. Thus, we can rewrite $(s, f(t)) \in \mathbf{S}^1_p \times \Gamma(\mathbf{S}^1_p, E)$ as $(s, f(s+\theta))$. This allows us to define the action of $h_1 = (\phi_1, g_1) \in \widehat{LG}$ on $\widehat{E}$ by,

$$(s, f(t)) \cdot h_1 = (s, f(s+\theta)) \cdot (\phi_1, g_1) \stackrel{\mathrm{def}}{=} (s+\phi_1, f(s+\theta) \cdot g_1(\theta)) \,. \tag{2.2}$$

To exhibit the group law in $\widehat{LG}$, we act again with $(\phi_2, g_2) \in \widehat{LG}$. In particular, if we define s' and θ' by the relations,

$$s' = s + \phi_1 \qquad \theta' = \theta - \phi_1 \,, \tag{2.3}$$

we find that

$$\begin{aligned} (s+\phi_1, f(s+\theta) \cdot g_1(\theta)) \cdot (\phi_2, g_2) &= (s', f(s'+\theta') \cdot g_1(\theta)) \cdot (\phi_2, g_2) && (2.4) \\ &= (s'+\phi_2, f(s'+\theta') \cdot g_1(\theta) \cdot g_2(\theta')) \,. && (2.5) \end{aligned}$$

From this we deduce that the group law for $\widehat{LG}$ must be,

$$(\phi_1, g_1) \cdot (\phi_2, g_2) = (\phi_1 + \phi_2, g_1(\theta) \cdot g_2(\theta')) = (\phi_1 + \phi_2, g_1(\theta) \cdot g_2(\theta - \phi_1)) \,, \tag{2.6}$$

which defines the semidirect product $\mathrm{U}(1) \ltimes LG$ where the action of U(1) on LG is given by rotation of the loop.

[4]More precisely, the problem is that $\mathbf{S}^1_p$ is a U(1) *torsor* rather than the group U(1) itself. The choice of an identity identifies the torsor with a copy of the group.

[5]We are indebted to Dan Freed for suggesting this.

We can summarize this construction as the following correspondence:

$$
\begin{tikzcd}
G \arrow[r] & E \arrow[d] \\
\mathrm{U}(1) \arrow[r] & Y \arrow[d] \\
& X
\end{tikzcd}
\quad \Longleftrightarrow \quad
\begin{tikzcd}
\widehat{LG} \arrow[r] & \widehat{E} \arrow[d] \\
& X
\end{tikzcd}
\tag{2.7}
$$

In fact, as we will show in section 4, this correspondence is invertible: given an $\widehat{LG}$ bundle, we can uniquely construct an E_8 bundle on the total space of its associated principal circle bundle.

3. The $\widehat{LE_8}$ bundle and its characteristic classes

As mentioned in the introduction, the particular case of $G = E_8$ is particularly relevant for M-theory. In light of the result of the previous section, it is interesting to ask if there might be a similarly fruitful relationship between $\widehat{LE_8}$ bundles on X_{10} and the antisymmetric tensor fields of type IIA. That such a relationship should exist is certainly not transparent from the string theory point of view as the quantization conditions for the RR forms of IIA are only known to be easily expressed in terms of classes in K-theory.

In this section, we will only begin to explore this question by asking if such a relationship can pass the most rudimentary test of consistency: do the characteristic classes of $\widehat{LE_8}$ bundles regarded as elements of de Rham cohomology obey the type IIA supergravity Bianchi identities? We will see that this is indeed the case. As further evidence for the relevance of loop groups of E_8 to IIA, we will also demonstrate that the characteristic classes of the centrally extended loop group are consistent with the Bianchi identities of massive IIA.

3.1 Supergravity and characteristic classes

Characteristic classes are certain elements in the cohomology of a manifold that characterize, often incompletely, principal bundles over that manifold. For example, for a $\mathrm{U}(N)$-bundle, these are the Chern classes. Or, for a U(1)-bundle, the sole characteristic class is the Euler class[6], e. As we will review in the next section, an E_8-bundle on Y_{11} is *completely* classified in low dimensions by a class in the fourth integer cohomology of the base. This is an analogous to the well known result that U(1)-bundles are classified by the Euler class. Making use of both these facts, the topological data in (2.1) is completely encoded in the following information:

$$
e \in H^2(X_{10}, \mathbb{Z}) \qquad \text{and} \qquad \mathrm{a} \in \mathrm{H}^4(\mathrm{Y}_{11}, \mathbb{Z})\,. \tag{3.1}
$$

[6]This is just the first Chern class in a flimsy disguise.

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Physically, these classes should correspond to topological fluxes in IIA and M-theory associated with G_2, the RR 2-form field strength in IIA string theory, and G, the M-theory 4-form. However, as described in [2], the correspondence is not quite what one might naively expect from (3.1). While it is true that $e = [G_2]$ in $H^2_{\mathrm{DR}}(X_{10})$, the correct quantization condition for the 4-form is $a = [G] - \frac{\lambda}{2} \in H^4(Y_{11}, \mathbb{Z})$ where $\lambda = p_1(Y_{11})/2$. This issue is properly dealt with in [4]. For the physical interpretation of what follows, we will assume that $p_1(Y_{11}) = 0$. Nonetheless, as there is a map to $\mathbf{B}E_8$ in [4], we believe that some variation of our construction will be relevant in the general case.[7]

In order to compare to IIA, however, we would like express the data of the E_8-bundle on Y_{11} in terms of geometric data on X_{10}.[8] As the results of the previous section imply that the data in (3.1) classify $\widehat{LE_8}$ bundles over X_{10} as well, one might guess that this information can be expressed in terms of the characteristic classes of the $\widehat{LE_8}$ bundle. To find the characteristic classes of the $\widehat{LE_8}$ bundle, we begin by asking if the data in (3.1) provide natural candidates for them. Of course, as $e = [G_2]$ is already expressed in terms of data on X_{10}, we can just reinterpret it as the characteristic class of the $\widehat{LE_8}$ bundle associated with the U(1) of $\widehat{LE_8}$. As for a, the only natural cohomology class on X_{10} that one can obtain from it is via the pushforward map which relates $H^4(Y_{11})$ to $H^3(X_{10})$. More explicitly, in terms of the M-theory four form G we have $H = \rho_*(G)$ where the pushforward map is given by integration over the $\mathbf{S}^1$ fiber, and H is interpreted as the NS 3-form field strength in IIA. In fact, there is an interesting interpretation of H (which we will return to later) as the obstruction to lifting the $\widehat{LE_8}$ bundle to a $\mathrm{U}(1) \ltimes \widetilde{LE}_8^{k=1}$ bundle.

As we will see in the next section, $[H]$ and $[G_2]$ are two characteristic classes of $\widehat{LE_8}$ bundles in ordinary cohomology in low dimensions. However, it turns out that these two classes are not independent as H is constrained by the fact that it arises as a pushforward of G. The nature of this constraint is best illustrated by the following simple example. Consider E_8-bundles over $Y = \mathbf{S}^3 \times \mathbf{S}^3$. Since $H^4(Y) = 0$ they must all be trivial. Now, consider the dimensional reduction of Y along the circle of a Hopf fibration of the first $\mathbf{S}^3$ factor to $X = \mathbf{S}^2 \times \mathbf{S}^3$. Clearly, there are nontrivial elements $[H] \in H^3(X)$, none of which are pushforwards of classes $[G] \in H^4(Y)$ in Y. Indeed, any G flux which has a nonzero pushforward, $H = \rho_*(G)$, would have to be nontrivial through a cycle of the form $\mathbf{S}^3 \times \mathbf{S}^1$ where $\mathbf{S}^1$ is the fiber. However, the Hopf fibration over $\mathbf{S}^2$ renders the cycle homologically trivial in Y. Now, as G_2 detects the Hopf fibration of the circle over the S^2, this situation is characterized by the fact that $\int_X H \wedge G_2 = \int_{\mathbf{S}^3} H \int_{\mathbf{S}^2} G_2 \neq 0$. Generalizing this example, if there exists a 5-cycle like X in X_{10} such that the H flux restricted to X is non-trivial

[7]In our picture, it is reasonable to conjecture that these gravitational corrections can be properly treated by adding the Spin(11) frame bundle of Y_{11} to the E_8 bundle and considering the dimensional reduction of the data of the full $G = E_8 \times \mathrm{Spin}(11)$ bundle. By the results of the previous section, this would suggest a description in IIA in terms of $\widehat{LG} = \mathrm{U}(1) \ltimes (LE_8 \times LSpin(11))$ bundles, where the U(1) acts by simultaneous rotation of the loop in both factors.

[8]As a classifies E_8-bundles, at the level of cohomology this problem is solved by a long exact sequence known as the Gysin sequence, as we will review in section 4. The loop group picture, however, provides a geometric interpretation for the result and generalizes to massive IIA.

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through a 3-cycle, and that the G_2 flux restricted to X is non-trivial through an intersecting 2-cycle so that $\int_X H \wedge G_2 \neq 0$, a similar argument would show that H does not descend from a G flux in Y_{11}. In fact, as we will show in section 4, the class $[H]$ is a pushforward of some $[G] \in Y_{11}$ if and only if $[H \wedge G_2] = 0$. Thus, the low dimensional characteristic classes of $\widehat{LE_8}$ obey $[H \wedge G_2] = 0$.

This presents us with a puzzle, however, if we were to hope that all the information about the E_8-bundle on Y_{11} would be contained in these characteristic classes. The pushforward ρ_* has a kernel including forms pulled back from the base. One can certainly have elements of $H^4(Y_{11})$ in that kernel. For example, consider an E_8-bundle over $Y = \mathbf{S}^1 \times \mathbf{S}^4$ which has n units of instanton number on the $\mathbf{S}^4$. Then, $[G]$ is just n times the unit volume form on $\mathbf{S}^4$. If we consider the dimensional reduction to $X = \mathbf{S}^4$, $H = \int_{\mathbf{S}^1} G = 0$ and so $[G]$ is clearly in the kernel of the pushforward map. Thus, the characteristic classes we have found will not detect that part of the classification of E_8 bundles. In other words, there are distinct E_8-bundles that give rise to identical $[H]$ and $[G_2]$, so these two characteristic classes cannot alone classify $\widehat{LE_8}$ bundles.

An obvious place to look for more information is the RR 4-form defined by

$$G_4 = \rho_*(\mathcal{A}\wedge G)\,, \tag{3.2}$$

where G is again the M-theory four form and we have introduced $\mathcal{A}$, an Ehresmann connection[9] on the U(1) bundle Y_{11}. While G_4 is gauge invariant, it is not closed and instead obeys a modified Bianchi identity:

$$dG_4 + G_2\wedge H = 0\,. \tag{3.3}$$

While the metric in M-theory does give us an explicit choice of a connection,[10] the $\widehat{LE_8}$ bundle is topological, and its characteristic classes should not depend on any such choice. We would like to find a mathematical context for understanding how G_4 might classify $\widehat{LE_8}$ bundles.

As we discuss in section 4, to do this we need to explore in greater detail the relationship between the cohomology ring of Y_{11} and that of X_{10}. The cohomology groups of Y_{11} and X_{10} fit into a cohomology long exact sequence known as the Gysin sequence.[11] In section 4 we will show how this sequence can be used to describe $a \in H^4(Y_{11}, \mathbb{Z})$ in terms of a class in a generalized cohomology theory on X_{10} which captures the information in both G_4 and H. While we will postpone a detailed treatment of this issue to that section, we will describe here the relevant results for a de Rham version of the theory.

We begin with a complex of differential forms[12]

$$\widehat{\Omega}^i(X_{10}) \stackrel{\text{def}}{=} \Omega^i(X_{10}) \oplus \Omega^{i-1}(X_{10})\,. \tag{3.4}$$

[9] This is also the global angular form of [24].

[10] In particular, if the metric on Y_{11} is $ds^2 = ds^2_{\text{base}} + (d\theta + \alpha)^2$, then $\mathcal{A} \propto d\theta + \alpha$.

[11] The Gysin sequence was used in a similar context in [10].

[12] We first encountered this complex in [25] which refers to [26]. All the proofs in this paper are due to us.

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Furthermore, we will choose an explicit representative of the Euler class, e, of the M-theory circle bundle. As above, this is just the RR 2-form. Given such a choice, G_2, we define the differential:

$$d(\alpha, \beta) = (d\alpha + G_2 \wedge \beta, -d\beta) \tag{3.5}$$

for $(\alpha, \beta) \in \widehat{\Omega}^i(X_{10})$. This squares to zero, so we can pass to the cohomology which we will denote $\widehat{H}^i_e(X_{10})$. It is not hard to see that the complexes for different choices of the representative of the Euler class are quasiisomorphic, i.e., have the same cohomology, justifying the notation. We will prove in the next section that the cohomology of this complex is, in fact, $H^4(Y_{11}, \mathbb{R})$.

(G_4, H) is closed in this complex as long as we have[13]

$$d\mathcal{A} = -\rho^* G_2 \qquad \text{and} \qquad \rho_*(\mathcal{A}) = 1\,. \tag{3.6}$$

Thus, it defines an element in the cohomology. One can verify that the class so obtained only depends on the cohomology class of G. We claim that this is the appropriate context for the IIA 4-form.

Given a cohomology class $[(G_4, H)]$, we can reconstruct the original class in $H^4_{\mathrm{DR}}(Y_{11})$ as follows. First, we must choose a connection. Then, we define the form:

$$G = \rho^*(G_4) - \mathcal{A} \wedge \rho^* H\,. \tag{3.7}$$

G is closed, so it defines a class in $H^4_{\mathrm{DR}}(Y_{11})$. It is an easy exercise to see that it does not depend on the addition of an exact 'form' to (G_4, H). Finally, let us show that it does not depend on the choice of $\mathcal{A}$. We choose two different connection $\mathcal{A}_1$ and $\mathcal{A}_2$. Then,

$$G^1 - G^2 = (\mathcal{A}_2 - \mathcal{A}_1) \wedge \rho^* H\,. \tag{3.8}$$

From (3.6), we have

$$d(\mathcal{A}_2 - \mathcal{A}_1) = -\rho^* G_2 + \rho^* G_2 = 0\,. \tag{3.9}$$

Now, because we have chosen a nontrivial circle bundle and a simply connected base, X_{10}, we have $H^1(Y_{11}, \mathbb{R}) = 0$. This can be seen, for example, from the Gysin sequence for the fibration. Since the cohomology is trivial, the difference in connections must be exact, i.e., $\mathcal{A}_2 - \mathcal{A}_1 = dB$, giving $G^1 - G^2 = d(B \wedge \rho^* H)$. This means that the cohomology class of G is independent of our choice of connection, and we are done.

Thus, we see that $e = [G_2] \in H^2(X_{10})$ and $[(G_4, H)] \in \widehat{H}^4_e(X_{10})$ are characteristic classes for the $\widehat{LE_8}$ bundle and contain all the information in the setup (2.1). The Bianchi identities for the RR two and four forms in type IIA supergravity follow from these definitions. This provides a mathematical context for these well-known equations.

[13] Note that these differ from the usual equations for a connection by a factor of 2π. We do this to save some space. The second condition is always possible because $d\rho_*(\mathcal{A}) = -\rho_* \rho^*(G_2) = 0$, so $\rho_*(\mathcal{A})$ is a constant function. Thus, the condition just reflects a choice of normalization.

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3.2 The Romans mass

Now, we turn to the situation where $G_0 = k$ is nonzero. This is the situation in massive IIA supergravity. Here, the Bianchi identity for G_2 is modified so that it obeys

$$dG_2 = G_0 H \tag{3.10}$$

in close analogy with the identity $dG_4 + G_2 \wedge H = 0$ we encountered above. Since G_2 is no longer a closed form, it cannot be the Euler class of a circle bundle, and the meaning of M-theory becomes somewhat unclear. It was conjectured in [1] that in this situation we should look at $\widetilde{LE}_8^k$ bundles over X_{10} where $\widetilde{LE}_8^k$ is the level k central extension of LE_8. Topologically, the group $LE_8 \cong E_8 \times \Omega E_8$, so $H^2(LE_8, \mathbb{Z}) = \mathbb{Z}$. Thus, there exist circle bundles over LE_8 labeled by the integers. The group $\widetilde{LE}_8^k$ is topologically the circle bundle over LE_8 with Euler class k times the generator of $H^2(LE_8, \mathbb{Z})$.

Using this fact, we can construct the first characteristic class of $\widetilde{LE}_8^k$ bundles. The homotopy exact sequence for the above circle fibration shows us that $\pi_1(\widetilde{LE}_8^k) = \mathbb{Z}_k$. Thus, we can use a simple obstruction theory argument[14] to give us a characteristic class $h \in H^2(X_{10}, \mathbb{Z}_k)$. Note that, since $\pi_1(\widetilde{LE}_8^{k=1}) = 0$, this class can be interpreted as the obstruction to lifting a given $\widetilde{LE}_8^k$ bundle to a bundle of its universal covering group, $\widetilde{LE}_8^{k=1}$, in analogy to the second Steifel-Whitney class for spin structures.[15] In order to motivate the relation of h to supergravity fields, first consider some cochain model $C^*(X_{10}, \mathbb{Z})$ for integral cohomology. The most obvious way to get an element $h \in H^2(X_{10}, \mathbb{Z}_k)$ in such a model is to take the mod k reduction of a closed cochain $\chi_2 \in C^2(X_{10}, \mathbb{Z})$ which represents a class in $H^2(X_{10}, \mathbb{Z})$. However, since we are working mod k, we really only need that χ_2 is closed mod k. That is, we only require that χ_2 obeys

$$d\chi_2 = k\chi_3\,, \tag{3.11}$$

for some cocycle χ_3. It can be shown that (3.11) defines a natural map $\beta : H^2(X_{10}, \mathbb{Z}_k) \to H^3(X_{10}, \mathbb{Z})$ by $\beta(h) = [\chi_3]$ which is called the Bockstein homomorphism. We will associate $\beta(h) = [H]$ with the quantized H-flux. Since $k\beta(h) = k[\chi_3] = [d\chi_2] = 0$, we see that $\beta(h) = [H]$ must correspond to a $\mathbb{Z}_k$ torsional class in $H^3(X, \mathbb{Z})$. In particular, this means that we can find a de Rham representative, kH, of $k\beta(h)$. Of course, de Rham cohomology does not detect torsion, so H, as a form, is exact even when not multiplied by k.[16] Further, note that the kernel of $\beta(h)$ consists precisely of mod k reductions of elements of $H^2(X_{10}, \mathbb{Z})$. We would like to associate this information with G_2. However, just as in the previous section, this is complicated by the modified Bianchi identity (3.10).

To find a model for G_2, we use the analogy between (3.11) and the modified Bianchi identity of G_4 in the previous section to construct a modified cochain complex for computing $H^i(X_{10}, \mathbb{Z}_k)$. The cochains consist of pairs (α, β) with $\alpha \in C^i(X_{10}, \mathbb{Z})$ and $\beta \in$

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[14] See, for example, [27].

[15] In fact [28], we can construct $\widetilde{LE}_8^k$ from $\widetilde{LE}_8^{k=1}$ by taking the quotient $\widetilde{LE}_8^k = \frac{\widetilde{LE}_8^{k=1} \times \mathrm{U}(1)}{\{(\theta, -k\theta) \in \mathrm{U}(1)_K \times \mathrm{U}(1)\}}$ where $\mathrm{U}(1)_K$ is the central element in $\widetilde{LE}_8^{k=1}$.

[16] Interestingly, torsion three forms have a simple geometric model in terms of $\mathrm{SU}(N)/\mathbb{Z}_N$ bundles or Azumaya algebras [29, 30].

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$C^{i+1}(X_{10}, \mathbb{Z})$, and the differential is defined by,

$$d(\alpha, \beta) = (d\alpha - k\beta, -d\beta)\,. \tag{3.12}$$

In analogy with the results described in the next section, the cohomology of this complex computes $H^i(X_{10}, \mathbb{Z}_k)$.[17] Now, comparing with (3.10) shows that we can interpret the pair (G_2, H) as a 2-cocycle in a de Rham version of the above complex. This de Rham version, of course, does not compute the torsional class, so we interpret the Bianchi identity as a reflection of the relation that holds in integral cohomology. Note that when $H = 0$, G_2 is closed and would naively represent an arbitrary element of integral cohomology. The quantization condition coming from $\widetilde{LE}_8^k$ suggests a nontrivial modification of this naive expectation, however. In fact, it is only the mod k reduction of this element of integral cohomology that is physically relevant. As we will discuss in section 5, this expectation is borne out by certain examples discussed in [21].

Massive IIA supergravity also has a four form which obeys the following Bianchi identity:

$$dG_4 = G_2 \wedge H = \frac{1}{G_0} G_2 \wedge dG_2\,. \tag{3.13}$$

This implies

$$\widetilde{G_4} = G_4 - \frac{1}{2G_0} G_2 \wedge G_2 \tag{3.14}$$

is closed. We believe that $\widetilde{G_4}$ should be thought of as a characteristic class for the $\widetilde{LE}_8^k$ bundle. The existence of such a class is proven in appendix A.

Thus, we have shown that the low dimensional characteristic classes of $\widetilde{LE}_8^k$ bundles encode the Bianchi identities of massive IIA. That loop group bundles seem to include the Romans mass (nonzero G_0) indicates that there may be a larger story to be told here. We hope to compute the full range of characteristic classes for $\widetilde{LE}_8^k$ bundles. This is still under investigation.

4. The cohomology of the classifying space

In this section, we will show how the $\widehat{LE}_8$-bundle constructed above fits into a larger structure of bundles and classifying spaces. We then show that the various classes introduced in the previous section are, in fact, characteristic classes of $\widehat{LE}_8$-bundles. We will also show that $\widehat{H}_e^*(X)$ computes the cohomology of the total space of the circle bundle Y.

[17] See footnote 18 for a short proof.

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4.1 A structure of classifying spaces

First, we will prove the existence and commutativity of the diagram:

$$\begin{tikzcd}
\mathrm{U}(1) \arrow[r] & \mathbf{B}LE_8 \arrow[d] \arrow[rr, bend left, "i"] & Y \arrow[l, "h"'] \arrow[r] \arrow[d, "\rho"] & \mathbf{B}E_8 \\
\mathbf{B}\,\mathrm{U}(1) & \mathbf{B}\widehat{LE_8} \arrow[l, "f"] & X \arrow[l, "g"] \arrow[ll, bend left]
\end{tikzcd} \quad . \tag{4.1}$$

In the process of doing so, we will encounter a number of other interesting constructions that shed some light on the central $\widehat{LE_8}$-bundle of this paper. In the above diagram, we begin with the space X and the circle bundle Y over X. This bundle is classified by the map from X to $\mathbf{B}\,\mathrm{U}(1)$. Over Y there exists an E_8-bundle classified by the map from Y to $\mathbf{B}E_8$. Any other Lie group can be substituted for E_8 in the above diagram, of course.

We begin by constructing the map h. Let $p \in Y$ be a point and let $\mathbf{S}^{\mathbf{1}}_p = \rho^{-1}\rho(p) \subset Y$ be the fiber containing p. The E_8-bundle, E over Y, is trivial over $\mathbf{S}^{\mathbf{1}}_p$, so the space of sections $\Gamma(\mathbf{S}^{\mathbf{1}}_p, E) \cong \mathrm{Maps}(\mathbf{S}^1, E_8)$. Assembling these for all points p, we see that there is a natural basepoint, given by p, and, thus, a global action by LE_8. This LE_8-bundle over Y, which we will denote E', defines the map h.

To construct i, we observe that LE_8 fits into the following exact sequence:

$$1 \longrightarrow \Omega E_8 \longrightarrow LE_8 \longrightarrow E_8 \longrightarrow 1\,. \tag{4.2}$$

The rightmost map in this sequence is the evaluation map $\mathrm{ev}_p : LE_8 \to E_8$. We can use this to associate an E_8-bundle to the LE_8-bundle as follows. Note that ΩE_8 is a normal subgroup of LE_8 and that the quotient is E_8. Thus, as a subgroup of LE_8, ΩE_8 acts on the LE_8 bundle. On each fiber, this action precisely preserves the E_8 over the basepoint in LE_8 which is simply the original E_8 over the point $p \in Y$. This means that the quotient is exactly the original E_8 bundle. In other words, the composition of the maps h and i in (4.1) is the map classifying the original E_8 bundle, E.

Now, let us examine the space E', the total space of the LE_8-bundle over Y. This space has a natural LE_8 action because it is a principal bundle. There is also a natural $\mathrm{U}(1)$ action defined by the simultaneous rotation of the circle in $\Gamma(\mathbf{S}^{\mathbf{1}}_p, E)$ and the circle in the fiber of Y over X. By composing the two projections, we can consider E' as a fiber bundle over X with fiber $\mathbf{S}^1 \times LE_8$. The LE_8 and $\mathrm{U}(1)$ actions just discussed combine to give a free $\widehat{LE_8}$ action on E'. In other words, E' is exactly the $\widehat{LE_8}$-bundle given by the construction in section 2.

Finally, $\widehat{LE_8}$ fits into the following exact sequence

$$1 \longrightarrow LE_8 \longrightarrow \widehat{LE_8} \longrightarrow \mathrm{U}(1) \longrightarrow 1\,. \tag{4.3}$$

In fact, this sequence splits on the right reflecting that $\widehat{LE_8}$ is a semidirect product. Now, let us consider the universal bundle, $\mathbf{E}\widehat{LE_8}$ over $\mathbf{B}\widehat{LE_8}$. As LE_8 is a subgroup of $\widehat{LE_8}$,

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it acts on $\mathbf{E}\widehat{LE_8}$, and we can take the quotient. As $\mathbf{E}\widehat{LE_8}$ is contractible and the action of LE_8 is free, the quotient is a $\mathbf{B}LE_8$. The remaining part of $\widehat{LE_8}$ is $\widehat{LE_8}/LE_8 \cong \mathrm{U}(1)$, so $\mathbf{B}LE_8$ is a U(1)-bundle over $\mathbf{B}\widehat{LE_8}$. By the above, the pullback of $\mathbf{E}\widehat{LE_8}$ by g is E'. Taking the quotient by LE_8 gives the space $\mathbf{B}LE_8$ and the map h by construction. The vertical lines in (4.1) are U(1) quotients. This implies that Y, as a circle bundle over X, is the pullback of the circle bundle $\mathbf{B}LE_8$ over $\mathbf{B}\widehat{LE_8}$. Thus, Y is classified by fg. This completes the proof that the diagram commutes.

We can now demonstrate how to invert the construction of section 2. Given an $\widehat{LE_8}$-bundle E' over X, we quotient by LE_8 to obtain Y and consider E' as a LE_8-bundle over Y. We can then recover the original E_8-bundle over Y as the associated bundle to this LE_8 bundle by the evaluation map as was done above.

4.2 The ordinary classes

Now, we turn to the characteristic classes. We begin by examining those that exist in ordinary cohomology. As stated above, the origin of the E_8 gauge field in M-theory is in the existence of a four-form field strength G whose quantization is controlled by $a \in H^4(Y, \mathbb{Z})$. Because its homotopy groups, $\pi_i(E_8)$, vanish for $i \neq 3$, $i \leq 15$, for the purposes of eleven dimensional manifolds, E_8 serves as a model for $K(\mathbb{Z}, 3)$. Since the classifying space functor shifts homotopy groups by one, this means that $\mathbf{B}E_8$ is a model for $K(\mathbb{Z}, 4)$. Finally, given that $H^4(Y, \mathbb{Z})$ is equivalent to the homotopy class of maps $[Y, K(\mathbb{Z}, 4)]$, we immediately see that E_8 bundles serve as models for elements in $H^4(Y, \mathbb{Z})$ and, hence, for the field strength G.

The rational cohomology of these Eilenberg-MacLane spaces is well known [24]. This gives us the rational cohomology of the classifying space $\mathbf{B}E_8$ in low dimensions. For dimensions less than twelve, there is only a degree four element and its square. The defining characteristic class of the E_8-bundle used in the previous section is the pull back of the class in $H^4(\mathbf{B}E_8, \mathbb{Z})$ which is, of course, the same class as in the previous paragraph. Thus, this class completely characterizes the E_8-bundle. One can also construct this class via obstruction theory [27] similarly to the class for $\widetilde{LE}^k_8$-bundles constructed in the previous section. Finally, if we were to choose a connection on the E_8-bundle, the four form can be given à la Chern-Weil as $\mathrm{Tr}(F \wedge F)$.

We compute the rational cohomologies of $\mathbf{B}LE_8$ and $\mathbf{B}\widehat{LE_8}$ in appendix A. The calculations involve the Leray-Serre spectral sequence applied to various fibrations obtainable from the exact sequences (4.2), (4.3). The structure in (4.1) also figures in prominently.

For $\mathbf{B}LE_8$, we obtain:

$$\begin{array}{r||c|c|c|c|c|c|c|c|c|c|c|c} i = & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \hline H^i(\mathbf{B}LE_8) & \mathbb{Q} & 0 & 0 & x & y & 0 & 0 & xy & y^2 & 0 & 0 & xy^2 \end{array} . \tag{4.4}$$

Note that $x^2 = 0$ as it is of odd degree.

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For $\mathbf{B}\widehat{LE_8}$, we obtain.

$$\begin{array}{c||c|c|c|c|c|c|c|c|c|c|c} i= & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline H^i(\mathbf{B}\widehat{LE_8}) & \mathbb{Q} & 0 & x & y & x^2 & 0 & x^3 & z & x^4 & 0 & x^6 \end{array} . \tag{4.5}$$

In the cohomology ring we have the relations $xy = xz = yz = 0$. Referring to (4.1), if w is the generator of the cohomology of $\mathbf{B}\,\mathrm{U}(1)$, then $x = f^*(w)$.

Also in (4.1), $g : X_{10} \to \mathbf{B}\widehat{LE_8}$ is a classifying map for the $\widehat{LE_8}$ bundle. We have the following identifications:

$$[G_2] = g^*(x) = (gf)^*(w) \qquad \text{and} \qquad [H] = g^*(y)\,. \tag{4.6}$$

The relation $xy = 0$ is reflected in the identity (3.3).

We also have the unfamiliar seven form z. Its proper interpretation, as we will demonstrate in the following subsection, is

$$2[H\wedge G_4] = g^*(z)\,. \tag{4.7}$$

This is a closed form, and it is not hard to see that it respects the relations $xz = yz = 0$ in the cohomology ring. The factor of two may seem mysterious, but it follows from the derivation (4.12). There are no other characteristic classes in dimensions less than eleven.

The commutativity of the diagram (4.1) tells us that the four form in M-theory is the pullback of the four form in $\mathbf{B}LE_8$. The spectral sequence in the appendix tells us that the pushforward of this form is the three form in $\mathbf{B}\widehat{LE_8}$. This accords with the fact that the three form in type IIA is the pushforward of the M-theory four form.

Finally, we note that there is an interesting local construction of H which elucidates its interpretation in terms of obstruction theory. To construct this class [31], choose a connection A on the $\widehat{LE_8}$ bundle $\pi : \widehat{E} \to X_{10}$. Let $L\mathfrak{g}$ and $\widetilde{L\mathfrak{g}}$ denote the Lie algebras of $\widehat{LE_8}$ and $\mathrm{U}(1) \ltimes \widetilde{LE_8}^{k=1}$ respectively. Consider an open set U in X_{10} over which we can locally lift the $\widehat{LE_8}$ bundle to a $\mathrm{U}(1) \ltimes \widetilde{LE_8}^{k=1}$ bundle. Such a lifting is a principal $\mathrm{U}(1) \ltimes \widetilde{LE_8}^{k=1}$-bundle $p : \tilde{E} \to U$ together with a $\widehat{LE_8}$ equivariant mapping $f : \tilde{E} \to \widehat{E}$ such that $p = \pi(f)$. Choose a connection A_U on $\tilde{E}$ which is compatible with A in the sense that $f^*A = q(A_U)$, where $q : \widetilde{L\mathfrak{g}} \to L\mathfrak{g}$ is the quotient map that annihilates the generator of the central element K. Since, $A_U \to A_U + \alpha K$ is still a compatible connection for any one form α on X_{10}, the space of such connections is an affine space under the space of one forms on X_{10}. Now, form the curvature $F_U = dA_U + A_U \wedge A_U$ and note that $F_U \to F_U + d\alpha K$. Define the scalar curvature Θ_U as the K component of F_U. We define $H_U = d\Theta_U$. Clearly, H_U is independent of the choice of connection A_U compatible with A. Thus, it patches together globally over any open cover of X_{10} to a closed 3-form H. Further, if this form is topologically nontrivial, the above local construction cannot be extended to the whole manifold, and therefore it measures the obstruction to lifting the $\widehat{LE_8}$ bundle to a $\mathrm{U}(1) \ltimes \widetilde{LE_8}^{k=1}$ bundle.

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4.3 The four form

Now, we would like to demonstrate a characteristic class in the cohomology of the complex given in (3.4). To do so, we will show that the complex computes the cohomology of the total space, Y, of a circle bundle over X with Euler class, e, as in figure 2. Using the four form in $H^4(\mathbf{B}LE_8)$ from (4.4), the fibration of $\mathbf{B}LE_8$ over $\mathbf{B}\widehat{LE_8}$ in (4.1) demonstrates the existence of the needed class. The commutativity of (4.1) demonstrates that the class does encode the correct E_8-bundle.

$$\begin{tikzcd} \mathrm{U}(1) \arrow[r] & Y \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

Figure 2: A circle bundle.

Rather than work with differential forms, we can define $\widehat{H}^*_e$ over the integers by replacing the forms with singular cochains. As such, for most of this section, we will consider cohomology with integer coefficients. The goal is to prove:

$$\widehat{H}^*_e(X,\mathbb{Z}) \cong H^*(Y,\mathbb{Z}) . \tag{4.8}$$

We will prove this isomorphism by exploiting the Gysin sequence in cohomology.[18] However, we would also like to show that the maps introduced in the previous section, (3.2) and (3.7), induce the above isomorphism when working with differential forms. This can be accomplished by exhibiting a chain homotopy which is the subject of appendix B.

Recall that the Gysin sequence is given by

$$\longrightarrow H^{i-2}(X) \xrightarrow{\cup e} H^i(X) \xrightarrow{\pi^*} H^i(Y) \xrightarrow{\pi_*} H^{i-1}(X) \xrightarrow{\cup e} H^{i+1}(X) \longrightarrow \tag{4.9}$$

This implies that $H^i(Y)$ can be decomposed as follows:

$$H^i(Y) \cong \pi^*\left[\mathrm{Coker}(\,\cup e) \subset H^i(X)\right] \oplus \pi_*^{-1}\left[\mathrm{Ker}(\,\cup e) \subset H^{i-1}(X)\right] . \tag{4.10}$$

To prove (4.8), we will show that $\widehat{H}^i_e(X)$ decomposes as in (4.10). We begin by determining the space of all possible inequivalent β for $[(\alpha,\beta)] \in \widehat{H}^i_e(X)$. We need that $d\beta = 0$. Furthermore, the invariance upon adding an exact form $d(a,b)$ means that we have $\beta \sim \beta - db$. In other words, the space of inequivalent β is a subgroup of $H^{i-1}(X)$. The condition that $d(\alpha,\beta) = 0$ also implies that for any cocycle representative G_2 of e, $d\alpha + G_2 \cup \beta = 0$. Therefore, we also require that $e \cup [\beta] = 0$. This tells us that the space of all possible β is exactly $\mathrm{Ker}(\,\cup e) \subset H^{i-1}(X)$.

Once we fix a β, we have to ask which α exist such that $[(\alpha,\beta)] \in \widehat{H}^i_e(X)$. As we have $e \cup [\beta] = 0$, we choose a γ such that $d\gamma + G_2 \cup \beta = 0$. Any element of $\widehat{H}^i_e(X)$ can now be written as $(\alpha,0) + (\gamma,\beta)$ with $d\alpha = 0$. The only remaining exact forms are those $d(a,b)$ with $db = 0$. This preserves the subgroup of pairs of the form $(\alpha,0)$. The invariance

[18] In fancy language, the proof is as follows. $\cup e$ is a chain map from $C^*(X,\mathbb{Z})$ to $C^{*+2}(X,\mathbb{Z})$. The cohomology of the mapping cone is then $\widehat{H}^*_e(X,\mathbb{Z})$. This fits into a long exact sequence in cohomology. Comparing this sequence with the Gysin sequence, the isomorphism (4.8) follows from the five-lemma. A similar argument applies to the complex (3.12) where the chain map is multiplication by k and we compare with the Bockstein long exact sequence.

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by adding $d(a,b)$ gives $\alpha \sim \alpha + da + e \cup b$ with $db = 0$. This tells us that the space of inequivalent α for a given $\beta \in \mathrm{Ker}(\cup e) \subset H^{i-1}(X)$ is exactly $\mathrm{Coker}(\cup e) \subset H^i(X)$ which demonstrates the decomposition (4.10).

One might note now that the cohomology of $\mathbf{B}LE_8$ has a degree three element and wonder how it appears in $\widehat{H}^3_e(X_{10})$. In fact, this three form, call it H, is just the pullback of the three form in $\mathbf{B}\widehat{LE_8}$. Hence, it maps to the class $(H,0)$ and is nothing new.

As a last exercise, we verify that the relation (4.7) is pulled back from the classifying space. For what follows, we will work with differential forms. Let $[(a,b)]$ be the element in $\widehat{H}^*_e(\mathbf{B}\widehat{LE_8})$ which pulls back to $[(G_4,H)]$. The form $c = 2(b\wedge a)$ is closed, and the addition of an exact 'form' to (a,b) adds an exact form to c, and, as such, does not affect its cohomology class. Finally, we need to demonstrate that $[c] = z$, the seven form in $H(\mathbf{B}\widehat{LE_8})$. In fact, for a general principal circle bundle with total space Y, base X and projection π, this construction gives a map $H^i(Y) \to H^{2i-1}(X)$ for i even. We claim that this map is

$$[D] \mapsto [\pi_*(D\wedge D)], \quad D \in \Omega^i(Y)\,. \tag{4.11}$$

Then, the claim that $[c] = z$ follows from the spectral sequence (A.6). There, the eight form in $H^*(\mathbf{B}LE_8)$ is the square of the four form and is given by θz in the E_2 term. The push forward maps this to z.

The addition of an exact form to D does not affect the cohomology class of $\rho_*(D\wedge D)$, so (4.11) is an operation on cohomology. By the considerations of appendix B, we know that, if $[D] \in H^*(Y)$ corresponds to $[(A,B)] \in \widehat{H}^*_e(X)$, then $[\rho^*(A) - \mathcal{A}\wedge\rho^*B] = [D]$. Squaring this and pushing forward, we obtain

$$\begin{aligned}
\pi_*([D]^2) &= [\pi_*\left(\pi^*A\wedge\pi^*A - 2\mathcal{A}\wedge\pi^*B\wedge\pi^*A\right)] \\
&= -2\left[\pi_*\left(\mathcal{A}\wedge\pi^*B\wedge\pi^*A\right)\right] \\
&= -(-1)^B 2\left[(B\wedge\pi_*(\mathcal{A}\wedge\pi^*A))\right] \\
&= 2\left[B\wedge A\right]
\end{aligned} \tag{4.12}$$

giving the desired answer.

5. Discussion

To conclude, we consider in greater detail the physical implications of the quantization conditions suggested by the loop group picture in IIA. We begin by considering the description of D8-branes in this context which requires an understanding of the relationship between $\widehat{LE_8}$ and $\widetilde{LE}_8^k$ more clearly. We then consider briefly the description of the other topological defects, i.e., D4-, NS5-, and D6-branes, in the loop group picture. In particular, we find that the mysterious torsional behavior of D6-brane charge in massive IIA has an interesting physical interpretation in the context of Calabi-Yau 3-fold compactifications of IIA with $G_0 \neq 0$ as studied in [21]. Furthermore, we consider an interesting example in which the K-theory description of the quantization condition at $g_s = 0$ disagrees with our picture and comment on the physical interpretation of the differences.

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5.1 D8-branes and loop groups

Regions of massive IIA arise in string theory separated by D8-brane domain walls. In the loop group picture, a D8-brane is a domain wall across which the fiber of the loop group bundle changes [16]. Thus, it is interesting to understand the relation between massive and massless IIA across such a domain wall. To approach this question, it is useful to compare the loop groups $\widehat{LE_8}$ and $\widetilde{LE}_8^k$. Topologically, $\widehat{LE_8}$ is $\mathbf{S}^1 \times LE_8$, which is the trivial bundle over LE_8. Thus, the various groups $\widehat{LE_8}$ and $\widetilde{LE}_8^k$ exhaust all U(1)-bundles over LE_8. There is a problem with this connection, however: the group structures of $\widehat{LE_8}$ and $\widetilde{LE}_8^k$ are very different. In particular, $\widehat{LE_8}$ fits into the exact sequence

$$1 \longrightarrow LE_8 \longrightarrow \widehat{LE_8} \longrightarrow \mathrm{U}(1) \longrightarrow 1\,, \tag{5.1}$$

while $\widetilde{LE}_8^k$ fits into the sequence

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \widetilde{LE}_8^k \longrightarrow LE_8 \longrightarrow 1\,. \tag{5.2}$$

This reflects the fact that $\widetilde{LE}_8^k$ is a central extension of LE_8 with U(1) as a normal subgroup while $\widehat{LE_8}$ is a semidirect product where U(1) is *not* a normal subgroup.[19] It would be interesting to understand if this picture of massive IIA is connected to the relations of massive IIA to M-theory conjectured by Hull [19] and further explored in [20, 18].

5.2 Branes in massless and massive IIA

It is clear that in passing from massive IIA to massless IIA, the geometric interpretation of the p-form field strengths is radically altered. Locally, we expect that the forms G_2, H, and G_4 are continuous across the domain wall. However, as their global structures are quite different, it is not clear how their associated topological charges, i.e., D6, NS5, and D4 brane charges, behave as we cross a D8-brane. Here, the geometric construction in terms of loop group bundles can be of some use.[20] In particular, the homotopy groups of the loop group fibers determine the possible topological defects. Using the fact that topologically in low dimensions,

$$\widehat{LE_8} \cong \mathbf{S}^1 \times LE_8 \cong \mathbf{S}^1 \times E_8 \times \Omega E_8 \approx K(\mathbb{Z},1) \times K(\mathbb{Z},2) \times K(\mathbb{Z},3) \tag{5.3}$$

we see that the low dimensional homotopy groups of $\widehat{LE_8}$ are just,

$$\pi_1\left(\widehat{LE_8}\right) = \pi_2\left(\widehat{LE_8}\right) = \pi_3\left(\widehat{LE_8}\right) = \mathbb{Z}\,. \tag{5.4}$$

[19] In fact, we can further form a semidirect product with U(1) for all the groups $\widetilde{LE}_8^k$. One might ask if the semidirect product circle is necessary in this situation. Since bundles of these groups would all have an additional 2-form characteristic class associated with the extra U(1), they do not appear to be relevant. Also, as we noted earlier, the group $\widehat{LE_8}$ can be thought of as a subgroup of $\mathrm{Diff}(\mathbf{S}^1) \ltimes LE_8$. This is an example of the Sugawara construction. In the centrally extended case, one can form the semidirect product using the connected part of $\mathrm{Diff}(\mathbf{S}^1)$. The significance of all of this is still obscure.

[20] Some of this analysis is also presented in [16].

This means that $\widehat{LE_8}$ bundles admit topological defects carrying $\mathbb{Z}$ valued charges of codimension 3, 4, and 5, corresponding to the D6, NS5, and D4 brane, as we would expect. Now, using the exact homotopy sequence of the fibration of (5.2), it is easy to see that the low dimensional homotopy groups of $\widetilde{LE}_8^k$ are,

$$\pi_1\left(\widetilde{LE}_8^k\right) = \mathbb{Z}_k \qquad \pi_2\left(\widetilde{LE}_8^k\right) = 0 \qquad \pi_3\left(\widetilde{LE}_8^k\right) = \mathbb{Z}\,. \tag{5.5}$$

Thus, we see that $\widetilde{LE}_8^k$ bundles admit a $\mathbb{Z}_k$ charged, codimension 3 topological defect which we might call a massive D6-brane, no defect corresponding to an NS5-brane, and a D4-brane. While it is natural to expect that D4-brane charge is identified across a D8-brane domain wall, the fate of the NS5-brane and D6-brane is less clear.

To address these questions, it is useful to recall the obstruction theory interpretation of some of the characteristic classes we have been discussing. We saw that $[H]$ in IIA can be identified as the obstruction to lifting the corresponding $\widehat{LE_8}$ bundle to a $\mathrm{U}(1) \ltimes \widetilde{LE}_8^{k=1}$ bundle. Since the NS5-brane sources an integral $[H]$ flux through a linking 3-cycle, this suggests that the associated $\widehat{LE_8}$ bundle cannot be lifted to a configuration in massive IIA with $G_0 = 1$. Thus, we expect that the NS5-brane cannot exist within a region of massive IIA. This is, of course, clearly the right answer as can be gleaned directly from the Bianchi identity of massive IIA, $dG_2 = G_0 H$.

The meaning of the $\mathbb{Z}_k$ charge and the fate of D6-branes is less obvious. As we noted earlier, in massive IIA, $[(G_2, H)]$ measures the obstruction to lifting a $\widetilde{LE}_8^k$ bundle to a $\widetilde{LE}_8^{k=1}$ bundle. Thus, this $\mathbb{Z}_k$ charge seems to be a defect that carries a nontrivial charge only if we have more than one D8-brane domain wall present, suggesting an interpretation in terms of the nonabelian gauge theory living on D8-branes. However, as the defect is of codimension 3, it also seems to be related to D6-branes which certainly need not be localized on a domain wall. It turns out [32, 33] that there exists a supergravity solution of massive IIA called the massive D6-brane which has been interpreted as a D8-D6-NS5 intersection, with the NS5-brane stuck to the domain wall. While this solution seems to be the kind of object we are looking for, it is not clear why these defects carry a discrete torsional charge from this perspective. Furthermore, the precise connection to D6-branes in massless IIA is still not obvious.[21]

While we will not be able to completely resolve this issue here, we will motivate an answer based on an interesting physical interpretation of the results of the previous paragraphs. It turns out that one can understand the behavior of NS5-branes and D6-branes in massive IIA via a kind of Stückelberg mechanism. In massive IIA, the gauge transformation of the NSNS B-field acts on the RR gauge potential C_1 as well,

$$\delta C_1 = -G_0 \Lambda_{NS} \qquad \delta B = d\Lambda_{NS}\,. \tag{5.6}$$

In particular, this suggests that we can locally gauge away the C_1 field in massive IIA. The interpretation (see, for example, [32]) for this is that the B-field actually becomes massive by eating C_1. In other words, C_1 is a Stückelberg field, and its kinetic energy term

[21] Other interesting bound states have been explored in [34, 35].

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becomes the mass term for B in unitary gauge. Under electric/magnetic duality, the dual RR C_7 field becomes massive by eating the dual NSNS B_6 field. Thus, we might expect that, at $g_s \neq 0$, regions of massive IIA repel magnetic B-fields, and, in particular, free magnetic monopoles such as NS5-branes would not be present. Furthermore, as the dual RR C_7 field is massive, one might expect that the vacuum of the theory has a condensate carrying its electric charge. In particular, this is the charge carried by a D6-brane, and if the condensate is a coherent state constructed out of quanta with charge k, this D6-brane charge is only conserved modulo k.

The work of Strominger and Polchinski [21] provides a concrete example of the above scenario. They consider Calabi-Yau compactifications of massive IIA down to four space-time dimensions. The key fact that we will need from their work is that in these compactifications the axio-dilaton (whose real part is related to the string coupling) acquires a nonzero RR 1-form magnetic charge of precisely G_0 units.[22] Thus, when $g_s \neq 0$ and therefore the axio-dilaton field has condensed, the magnetic charge it carries (which is precisely a D6-brane charge) is only conserved modulo G_0. In fact, they show that near the conifold point, this behavior can be attributed to the condensation of D6-branes wrapped on the CY 3-fold. Note, however, that this interpretation fails precisely when $g_s = 0$ making D6-brane charge $\mathbb{Z}$ valued as one might expect from K-theory considerations. We believe that this lends support to the notion that the $\widetilde{LE}_8^k$ quantization condition should be interpreted as an important ingredient in describing a kind of M-theoretic, finite g_s dual to massive IIA.

5.3 Loop groups and K-theory

Finally, we would like to make a more direct comparison between the loop group picture and K-theory on certain simple manifolds. Consider type IIA compactified to 6-dimensions on $X = \mathbb{C}\mathbf{P}^1 \times \mathbb{C}\mathbf{P}^1$ (if we wish to, we can further compactify the remaining six dimensions on a six-manifold with positive scalar curvature, for example, a six-sphere). We will be interested in analyzing nontrivial G_2 and G_4 flux configurations on X corresponding to the K-theory classes of the the virtual bundles, $x_n = \mathcal{O}(n,n) \ominus \mathcal{O}(n-1,n-1)$. As X and all its factors have trivial normal bundle and positive scalar curvature, the arguments of section 4 of [3] show that there is no nontrivial shift in the quantization conditions. Furthermore, as these virtual bundles have rank 0, if x and y are the generators of the cohomology of the two $\mathbb{C}\mathbf{P}^1$ fibers, we have,

$$\begin{aligned} [G(x_n)] &= [G_2] + [G_4] = \sqrt{\hat{A}(X)}ch(x_n) = ch(x_n) \\ &= e^{n(x+y)} - e^{(n-1)(x+y)} = (x+y) + \left(n - \frac{1}{2}\right)(x+y)^2 \,. \end{aligned} \tag{5.7}$$

As their Chern characters are clearly different, the classes x_n are distinguishable and correspond to distinct allowed flux configurations on X. What is interesting in this example is that, as n varies, the K-theory classes have the same $[G_2]$ while their values of $[G_4]$ differ by integral multiples of $[\alpha \wedge G_2]$ for some closed α (here, $\alpha = G_2$). In particular, their

[22] This connection was actually used to justify the quantization condition on G_0.

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differences are trivial as elements of $\widehat{H}_e^4(X)$ and lift to the same class a in M-theory. This is, then, a simple example in which the K-theory description of the quantization of fluxes seems to be at odds with that of M-theory even in the absence of torsion. Note, however, that while the topological data of the E_8-bundle in M-theory is insensitive to the difference between these configurations, the same is not true for the CS terms in the M-theory action. Thus, this data is certainly detected by the η invariant of the E_8-bundle. While we will not be able to resolve this puzzle, we can see two possible resolutions. The authors of [7] were able to escape this difficulty by considering their E_8-bundles as pulled back from the ten dimensional base, giving a finer characterization. This suggests that there may be some equivariant information that could distinguish E_8-bundles. Another possibility is that, as we saw in the previous example, it is possible that finite g_s effects, perhaps non-perturbative instanton effects, may be responsible for a breakdown in the K-theory picture.

Let us end by briefly touching on the issue of twisted K-theory. We can combine the form fields present in supergravity with their Hodge duals to generate the self-dual form:

$$G_t = G_0 + G_2 + G_4 + \star G_4 + \star G_2 + \star G_0 \,. \tag{5.8}$$

The Bianchi identities and the equations of motion imply that

$$dG_t + H \wedge G_t = 0 \,. \tag{5.9}$$

If we define an operation on an arbitrary form, F, by

$$d_H(F) = dF + H \wedge F \,, \tag{5.10}$$

we have $d_H^2 = 0$ and can take its cohomology. This is sometimes termed *twisted cohomology* and is denoted $H(X_{10}, [H])$ [36]. This cohomology is tantalizingly similar, but in many ways opposite, to the generalized cohomology defined in the text.

It is believed that RR forms in IIA, rather than being elements in this cohomology, however, actually live in the twisted K-theory group $K^0_{[H]}(X_{10})$ [9, 37]. There is a Chern character that maps this group to the twisted cohomology defined above [38]. The relation between twisted cohomology, twisted K-theory and E_8 gauge theory has been explored in [10, 12].

If it turns out that $\widehat{LE_8}$ bundles fully capture the information of the 4-form in M-theory, we would have two alternate quantizations of the fields of type IIA supergravity. One might hope that this could shed some light on the calculations of [7].

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A. Some cohomology computations

A.1 $\mathbf{B}LE_8$

First, we compute the cohomology of $\mathbf{B}LE_8$. We recall the exact sequence

$$1 \longrightarrow \Omega E_8 \longrightarrow LE_8 \longrightarrow E_8 \longrightarrow 1 \tag{A.1}$$

where the last map is given by evaluation at the basepoint of $\mathbf{S}^1$. Similarly to the construction of the circle bundle over $\mathbf{B}\widehat{LE_8}$ in section 4.1, we can form the principal bundle:

$$\begin{tikzcd} E_8 \arrow[r] & \mathbf{B}\Omega E_8 \arrow[d] \\ & \mathbf{B}LE_8\,. \end{tikzcd} \tag{A.2}$$

Now, we can look at the Leray-Serre spectral sequence[23] for (A.2). The Whitehead theorem tells us that the cohomology of both $\mathbf{B}\Omega E_8$ and E_8 is that of $K(\mathbb{Z}, 3)$ in low dimensions. If we denote the generator of $H^3(K(\mathbb{Z}, 3), \mathbb{Z})$ by θ, the only choice we have is whether or not $d_4\theta = 0$. As the Hurewicz isomorphism tells us that $H^3(\mathbf{B}LE_8, \mathbb{Z}) = \mathbb{Z}$, we must kill θ, implying the existence of an element y such that $d_4\theta = y$. Thus, we can write down the E_2 ($= E_3 = E_4$) term as

3	θ			θx	θy			θxy	θy^2			θxy^2
2												
1												
0	$\mathbb{Z}$			x	y			xy	y^2			xy^2
	0	1	2	3	4	5	6	7	8	9	10	11

. (A.3)

After we take the cohomology of d_4 to obtain E_5, we see that the spectral sequence collapses, at least for the low dimensions that we care about. The only terms that survive are $\mathbb{Z}$ and x giving the rational cohomology of $K(\mathbb{Z}, 3)$. The zero row of (A.3) is the cohomology of $\mathbf{B}LE_8$ yielding the result (4.4). In fact, one can see directly that $\mathbf{B}LE_8$ is approximately $K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 4)$ in low dimensions, but we will not do so here.

A.2 $\mathbf{B}\widehat{LE_8}$

We now compute the cohomology of $\mathbf{B}\widehat{LE_8}$. We begin by recalling the diagram 4.1:

$$\begin{tikzcd} \mathrm{U}(1) \arrow[r] & \mathbf{B}LE_8 \arrow[d] \arrow[rr, bend left, "i"] & Y \arrow[l] \arrow[r] \arrow[d] & \mathbf{B}E_8 \\ \mathbf{B}\,\mathrm{U}(1) & \mathbf{B}\widehat{LE_8} \arrow[l] & X \arrow[l] & \end{tikzcd}\ . \tag{A.4}$$

[23] See, for example, [24, 39].

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First, note that there exists a two form pulled back from $\mathbf{B}\,\mathrm{U}(1)$. As any circle bundle can exist as a part of the $\widehat{LE_8}$-bundle, any map to $\mathbf{B}\,\mathrm{U}(1)$ factors through $\mathbf{B}\widehat{LE_8}$. This reflects that the sequence (4.3) splits. This universality implies that no power of the two form in $\mathbf{B}\widehat{LE_8}$ can vanish.

Next, we isolate the following fibration from (A.4):

$$\begin{array}{ccc} \mathrm{U}(1) & \longrightarrow & \mathbf{B}LE_8 \\ & & \downarrow \\ & & \mathbf{B}\widehat{LE_8} \end{array} \quad . \tag{A.5}$$

When constructing the E_2 term of the associated spectral sequence, there are two possibilities. The question is whether the four form of $\mathbf{B}LE_8$ is pulled back from $\mathbf{B}\widehat{LE_8}$ or not. We will see that it is not. Thus, the E_2 term is

$$\begin{array}{c||c|c|c|c|c|c|c|c|c|c|c} 1 & \theta & & \theta x & \theta y & \theta x^2 & & \theta x^3 & \theta z & \theta x^4 & & \theta x^6 \\ \hline 0 & \mathbb{Z} & & x & y & x^2 & & x^3 & z & x^4 & & x^6 \\ \hline\hline & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \tag{A.6}$$

where all products of x, y and z are zero. As above, the zero row gives the cohomology (4.5). Note that the four form in $\mathbf{B}LE_8$ is θy which pushes forward to y, the three form in $\mathbf{B}\widehat{LE_8}$.

In order to prove the result about the four form, we proceed by examining the following fibration which follows from (4.2,A.1):

$$\begin{array}{ccc} \mathbf{B}\Omega E_8 & \longrightarrow & \mathbf{B}LE_8 \\ & & \downarrow{\scriptstyle i} \\ & & \mathbf{B}E_8 \end{array} \quad . \tag{A.7}$$

The E_2 term in the associated spectral sequence is as follows:

$$\begin{array}{c||c|c|c|c|c} 3 & \theta & & & & \theta x \\ \hline 2 & & & & & \\ \hline 1 & & & & & \\ \hline 0 & \mathbb{Z} & & & & x \\ \hline\hline & 0 & 1 & 2 & 3 & 4 \end{array} \quad . \tag{A.8}$$

As we know the total cohomology already from the previous section, we see that there are no nontrivial differentials. This implies that the four form in $\mathbf{B}LE_8$ is pulled back from the four form in $\mathbf{B}E_8$ by i.

Any four form in Y can be represented by a map to $\mathbf{B}E_8$. The commutativity of (A.4) implies that this four form can be thought of as pulled back from $\mathbf{B}LE_8$. Now, we assume,

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as above, that the four form in $\mathbf{B}LE_8$ is *also* pulled back from $\mathbf{B}\widehat{LE_8}$. By commutativity, we could then pull it back to Y through X, giving the same form as when pulled back from $\mathbf{B}E_8$. Now, let $Y \cong \mathbf{S}^3 \times \mathbf{S}^1$ and $X \cong \mathbf{S}^3$. There exists a nontrivial four form on Y which clearly cannot be pulled back from X. Thus, we have a contradiction, so the four form on $\mathbf{B}LE_8$ is not pulled back from $\mathbf{B}\widehat{LE_8}$.

A.3 $\mathbf{B}\widetilde{LE}_8^k$

As a last exercise, we prove the existence of a four form characteristic class in $\mathbf{B}\widetilde{LE}_8^k$. First, we recall that $\widetilde{LE}_8^k$ fits into the following fibration:

$$\begin{array}{ccc} \mathrm{U}(1) & \longrightarrow & \widetilde{LE}_8^k \\ & & \big\downarrow \\ & & LE_8 \end{array} . \tag{A.9}$$

In low dimensions, the homotopy sequence of the fibration gives the following set of homotopy groups for $\widetilde{LE}_8^k$:

$$\begin{array}{r||c|c|c|c|c|c|c} i = & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline \pi_i(\widetilde{LE}_8^k) & \mathbb{Z} & \mathbb{Z}_p & 0 & \mathbb{Z} & 0 & 0 & 0 \end{array} . \tag{A.10}$$

In fact [28], for $k = 1$, the group is simply connected, and for higher k, $p = k$. The higher extensions are all $\mathbb{Z}_k$ quotients of the level one extension. Finally, it follows from the rational Hurewicz isomorphism that $H^4(\mathbf{B}\widetilde{LE}_8^k, \mathbb{Q}) = \mathbb{Q}$.

B. The chain homotopy

In this appendix, we will show that the equations (3.2) and (3.7) induce an isomorphism between $\widehat{H}_e^*(X) \cong H^*(Y)$. First, we examine the following diagram:

$$\begin{array}{ccccccc} \longrightarrow & \Omega^i(Y) & \xrightarrow{d} & \Omega^{i+1}(Y) & \longrightarrow \\ & j^{-1}\big\uparrow\big\downarrow j & & j^{-1}\big\uparrow\big\downarrow j & \\ \longrightarrow & \widehat{\Omega}^i(X) & \xrightarrow{d} & \widehat{\Omega}^{i+1}(X) & \longrightarrow \end{array} \tag{B.1}$$

where $j(\alpha) = (-1)^\alpha(\rho_*(\mathcal{A}\wedge\alpha), \rho_*(\alpha))$ and $j^{-1}(\beta,\gamma) = \rho^*(\beta) - \mathcal{A}\wedge\rho^*(\gamma)$, essentially the maps (3.2) and (3.7). In $(-1)^\alpha$, α is just the degree of the form α and the multiplication is distributive across the direct sum. It is easy to verify that j and j^{-1} are chain maps. As always, we choose a specific representative of the Euler class, e, and a connection, $\mathcal{A}$, that satisfy the relations (3.6).

Next, we must show that the induced maps on cohomology are actually isomorphisms. First, we show that $j \circ j^{-1}$ is the identity on the level of forms:

$$(j \circ j^{-1})(\beta, \gamma) = j(\rho^*(\beta) - \mathcal{A}\wedge\rho^*(\gamma))$$

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$$
\begin{aligned}
&= (-1)^{\beta}(\rho_*(\mathcal{A}\wedge(\rho^*(\beta) - \mathcal{A}\wedge\mathcal{A}\wedge\rho^*(\gamma))), \rho_*(\rho^*(\beta) - \mathcal{A}\wedge\rho^*(\gamma))) \\
&= (\rho_*(\mathcal{A})\cdot\beta, \rho_*\rho^*(\beta) + \rho_*(\mathcal{A})\cdot\gamma) \\
&= (\beta,\gamma)\,. \qquad \text{(B.2)}
\end{aligned}
$$

Now, let us examine the situation for $j^{-1}\circ j$:

$$
\begin{aligned}
(j^{-1}\circ j)(\alpha) &= (-1)^{\alpha} j^{-1}(\rho_*(\mathcal{A}\wedge\alpha), \rho_*(\alpha)) \\
&= (-1)^{\alpha}\left[\rho^*\rho_*(\mathcal{A}\wedge\alpha) - \mathcal{A}\wedge\rho^*\rho_*(\alpha)\right]. \qquad \text{(B.3)}
\end{aligned}
$$

This is clearly not equal to α. In order to go further, we must investigate the properties of the operator $\rho^*\rho_*$. Let us write α in local coordinates as

$$
\alpha = f^i(\theta,\mathbf{x})d\mathbf{y}_i\wedge d\theta + g^i(\theta,\mathbf{x})d\mathbf{z}_i\,, \qquad \text{(B.4)}
$$

where $d\mathbf{y}_i$ and $d\mathbf{z}_i$ are bases of forms of the appropriate degree on the base pulled back by the fiber projection. The sums over i are understood. Then, we have:

$$
\rho^*\rho_*(\alpha) = \left(\int d\theta f^i(\theta,\mathbf{x})\right)d\mathbf{y}_i\,. \qquad \text{(B.5)}
$$

We can also write $\mathcal{A}$ in local coordinates as:

$$
\mathcal{A} = \frac{1}{2\pi}d\theta + \rho^*(\psi)\,. \qquad \text{(B.6)}
$$

This follows because the connection is invariant with respect to the U(1) action on the fiber implying that there cannot be any θ dependence in the coefficient of $d\theta$ or in the basic form $\rho^*(\psi)$. Furthermore the normalization condition in (3.6) forces the coefficient of $d\theta$ to be $1/2\pi$. We also note that (B.3) is invariant under additions to $\mathcal{A}$ of the form $\rho^*(\psi)$ so we can neglect that term. Finally, (B.3) is linear on α and vanishes for $\alpha = \rho^*(\beta)$.

We can put all this together to obtain:

$$
(j^{-1}\circ j)(\alpha) = \left(\frac{1}{2\pi}\int d\theta g^i\right)d\mathbf{z}_i + \left(\frac{1}{2\pi}\int d\theta f^i\right)d\mathbf{y}_i\wedge d\theta\,. \qquad \text{(B.7)}
$$

Again, from the linearity of (B.3), we can deal with the terms in (B.4) separately. Let us define an operator, K, as follows:

$$
\begin{aligned}
K(f^i(\theta,\mathbf{x})d\mathbf{y}_i\wedge d\theta) &= \tilde{f}^i d\mathbf{y}_i \\
K(g^i(\theta,\mathbf{x})d\mathbf{z}_i) &= 0\,. \qquad \text{(B.8)}
\end{aligned}
$$

Here $\tilde{f}^i$ is the function defined such that:

$$
\frac{\partial\tilde{f}^i}{\partial\theta} = f^i - \frac{1}{2\pi}\int d\theta f^i \qquad \text{(B.9)}
$$

and

$$
\int d\theta\tilde{f}^i = 0 \qquad \text{(B.10)}
$$

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which can always be achieved by adjusting the constant of integration.

Finally, we look at $(dK - Kd)\alpha$:

$$
\begin{aligned}
(dK - Kd)\alpha &= d(\tilde{f} d\mathbf{y}_i) - K\left[\left(\frac{\partial \tilde{f}^i}{\partial \mathbf{x}_j} d\mathbf{x}_j\right) \wedge d\mathbf{y}_i \wedge d\theta + (-1)^\alpha \frac{\partial g^i}{\partial \theta} d\mathbf{z}_i \wedge d\theta + \right.\\
&\qquad \left. + \left(\frac{\partial g^i}{\partial \mathbf{x}_j} d\mathbf{x}_j\right) \wedge d\mathbf{z}_i\right] \\
&= \left[\frac{\partial \tilde{f}}{\partial \mathbf{x}_j} d\mathbf{x}_j \wedge d\mathbf{y}_i + (-1)^{\alpha-1}\left(f^i - \frac{1}{2\pi}\int d\theta f^i\right) d\mathbf{y}_i \wedge d\theta\right] - \\
&\quad - \widetilde{\frac{\partial f^i}{\partial \mathbf{x}_j}} d\mathbf{x}_j \wedge d\mathbf{y}_i - (-1)^\alpha \widetilde{\frac{\partial g^i}{\partial \theta}} d\mathbf{z}_i \\
&= (-1)^{\alpha-1}\left[\left(f^i - \frac{1}{2\pi}\int d\theta f^i\right) d\mathbf{y}_i d\theta + \left(g_i - \frac{1}{2\pi}\int d\theta g_i\right) d\mathbf{z}_i\right] \\
&= (-1)^{\alpha-1}(1 - (j^{-1} \circ j)(\alpha)) \,.
\end{aligned} \tag{B.11}
$$

where we have used the fact that the tilde operation commutes with differentiating on variables that are not θ and the following:

$$
\frac{\partial}{\partial \theta} \widetilde{\frac{\partial g^i}{\partial \theta}} = \frac{\partial g^i}{\partial \theta} \Longrightarrow \widetilde{\frac{\partial g^i}{\partial \theta}} = g^i - \int d\theta g^i \,. \tag{B.12}
$$

This shows that K is a homotopy operator, and therefore j and j^{-1} do induce isomorphisms in cohomology.

References

[1] A. Adams and J. Evslin, *The loop group of E_8 and K-theory from 11D*, *JHEP* **02** (2003) 029 [hep-th/0203218].

[2] E. Witten, *On flux quantization in M-theory and the effective action*, *J. Geom. Phys.* **22** (1997) 1 [hep-th/9609122].

[3] G.W. Moore and E. Witten, *Self-duality, Ramond-Ramond fields and K-theory*, *JHEP* **05** (2000) 032 [hep-th/9912279].

[4] E. Diaconescu, G.W. Moore and D.S. Freed, *The M-theory 3-form and E_8 gauge theory*, hep-th/0312069.

[5] P. Hořava and E. Witten, *Heterotic and type-I string dynamics from eleven dimensions*, *Nucl. Phys.* **B 460** (1996) 506 [hep-th/9510209].

[6] P. Hořava and E. Witten, *Eleven-dimensional supergravity on a manifold with boundary*, *Nucl. Phys.* **B 475** (1996) 94 [hep-th/9603142].

[7] D.-E. Diaconescu, G.W. Moore and E. Witten, *E_8 gauge theory and a derivation of K-theory from M-theory*, *Adv. Theor. Math. Phys.* **6** (2003) 1031 [hep-th/0005090].

[8] P. Horava, unpublished.

[9] P. Bouwknegt and V. Mathai, *D-branes, B-fields and twisted k-theory*, *JHEP* **03** (2000) 007 [hep-th/0002023].

JHEP06(2005)043

[10] V. Mathai and H. Sati, *Some relations between twisted k-theory and E_8 gauge theory*, *JHEP* **03** (2004) 016 [hep-th/0312033].

[11] J. Evslin, *From E_8 to F via T*, *JHEP* **08** (2004) 021 [hep-th/0311235].

[12] J. Evslin, *Twisted k-theory from monodromies*, *JHEP* **05** (2003) 030 [hep-th/0302081].

[13] J. Evslin, *A torsion correction to the RR 4-form fieldstrength*, *JHEP* **11** (2003) 037 [hep-th/0304098].

[14] P. Bouwknegt, J. Evslin and V. Mathai, *T-duality: topology change from H-flux*, *Commun. Math. Phys.* **249** (2004) 383 [hep-th/0306062].

[15] I. Kriz and H. Sati, *M-theory, type-IIA superstrings and elliptic cohomology*, *Adv. Theor. Math. Phys.* **8** (2004) 345 [hep-th/0404013].

[16] A. Adams, J. Evslin, and U. Varadarajan, *G_0 from E_8*, to appear.

[17] L.J. Romans, *Massive N=2A supergravity in ten-dimensions*, *Phys. Lett.* **B 169** (1986) 374.

[18] G.W. Moore and N. Saulina, *T-duality and the K-theoretic partition function of type IIA superstring theory*, *Nucl. Phys.* **B 670** (2003) 27 [hep-th/0206092].

[19] C.M. Hull, *Massive string theories from M-theory and F-theory*, *JHEP* **11** (1998) 027 [hep-th/9811021].

[20] M. Haack, J. Louis and H. Singh, *Massive type-IIA theory on K3*, *JHEP* **04** (2001) 040 [hep-th/0102110].

[21] J. Polchinski and A. Strominger, *New vacua for type-II string theory*, *Phys. Lett.* **B 388** (1996) 736 [hep-th/9510227].

[22] L. Dolan and M.J. Duff, *Kac-Moody symmetries of Kaluza-Klein theories*, *Phys. Rev. Lett.* **52** (1984) 14.

[23] L. Dolan, *Symmetries of massive fields in Kaluza-Klein supergravity*, *Phys. Rev.* **D 30** (1984) 2474.

[24] R. Bott and L.W. Tu, *Differential Forms in Algebraic Topology*, vol. 82 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 1982.

[25] J.I. Royo Prieto, *The Gysin sequence for riemannian flows*, in *Global differential geometry: the mathematical legacy of Alfred Gray, Bilbao, 2000*, vol. 288 of *Contemp. Math.*, pp. 415–419, *Amer. Math. Soc.*, Providence, RI, 2001.

[26] F.W. Kamber and P. Tondeur, *Duality theorems for foliations*, *Astérisque*, 1984, no. 116, 108–116, *Transversal structure of foliations Toulouse*, 1982.

[27] E. Witten, *Topological tools in ten-dimensional physics*, *Int. J. Mod. Phys.* **A 1** (1986) 39.

[28] A. Pressley and G. Segal, *Loop Groups*. Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1986, Oxford Science Publications.

[29] E. Witten, *D-branes and k-theory*, *JHEP* **12** (1998) 019 [hep-th/9810188].

[30] A. Kapustin, *D-branes in a topologically nontrivial B-field*, *Adv. Theor. Math. Phys.* **4** (2000) 127 [hep-th/9909089].

[31] J.L. Brylinski, *Loop spaces, characteristic classes and geometric quantization*, vol. 107 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1993.

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[32] B. Janssen, P. Meessen and T. Ortin, *The D8-brane tied up: string and brane solutions in massive type-IIA supergravity*, *Phys. Lett.* **B 453** (1999) 229 [hep-th/9901078].

[33] Y. Imamura, *1/4 BPS solutions in massive IIA supergravity*, *Prog. Theor. Phys.* **106** (2001) 653 [hep-th/0105263].

[34] H. Singh, *Duality symmetric massive type-II theories in $D = 8$ and $D = 6$*, *JHEP* **04** (2002) 017 [hep-th/0109147].

[35] H. Singh, *Note on (D6,D8) bound state, massive duality and non- commutativity*, *Nucl. Phys.* **B 661** (2003) 394 [hep-th/0212103].

[36] R. Rohm and E. Witten, *The antisymmetric tensor field in superstring theory*, *Ann. Phys. (NY)* **170** (1986) 454.

[37] P. Bouwknegt, A.L. Carey, V. Mathai, M.K. Murray and D. Stevenson, *Twisted K-theory and K-theory of bundle gerbes*, *Commun. Math. Phys.* **228** (2002) 17 [hep-th/0106194].

[38] V. Mathai and D. Stevenson, *Chern character in twisted K-theory: equivariant and holomorphic cases*, *Commun. Math. Phys.* **236** (2003) 161 [hep-th/0201010].

[39] J. McCleary, *A user's guide to spectral sequences*, vol. 58 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, second ed., 2001.

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